Exponentiation of certain Matrices related to the Four Level System by use of the Magic Matrix

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6 March 2007

Abstract

In this paper we show how to calculate explicitly the exponential of certain matrices, which are evolution operators governing the interaction of the four level system of atoms and the radiation, etc. We present a consistent method in terms of the magic matrix by Makhlin.

As a closely related subject, we derive a closed form expression of the Baker-Campbell-Hausdorff formula for a class of matrices in $SU(4)$, by use of the method developed by the present authors in quant-ph/0610009.

1 Introduction

The purpose of this paper is to develop a useful method to calculate the exponential of certain matrices explicitly. These matrices arise from differential equations governing the interaction of the four level system of atoms with the radiation whose image is illustrated in Figure 1.

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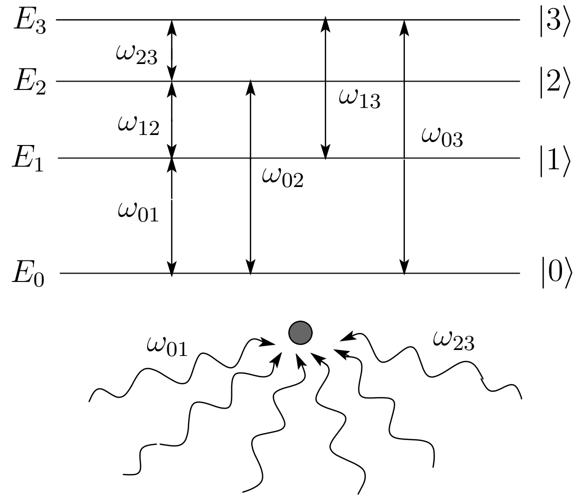

Fig.1 Atom with four energy levels and general action by laser fields

For the general background, motivation and possible applications of the present work, see the recent work of the author and collaborators [1], [2] and [3]. Under the RWA (Rotating Wave Approximation) and some resonance conditions the problem is reduced to the evaluation of the exponential (i.e. finite time evolution operator) e^{-itH} with the Hamiltonian

$$H = \begin{pmatrix} 0 & h_{12} & h_{13} & h_{14} \\ h_{12} & 0 & h_{23} & h_{24} \\ h_{13} & h_{23} & 0 & h_{34} \\ h_{14} & h_{24} & h_{34} & 0 \end{pmatrix},$$

where h_{ij} are real coupling constants between the atom and laser fields. Here we have changed some notation from the previous one in [2] (like $g_{ij} \leftrightarrow h_{i-1,j-1}$) for convenience. A generic algorithm to calculate e^{-itH} based on eigenvalues of the Hamiltonian H has been given by us in [2], although the actual execution is not so simple.

In this paper we revisit the problem from a different point of view. Let us introduce Makhlin's theorem. The isomorphism

$$SU(2) \otimes SU(2) \cong SO(4) \quad (\Longleftrightarrow su(2) \otimes 1_2 + 1_2 \otimes su(2) \cong so(4))$$

is one of the well-known theorems in elementary representation theory and is characteristic of four dimensional Euclidean space. In [4] Makhlin gave it the adjoint expression explicitly:

$$F : SU(2) \otimes SU(2) \longrightarrow SO(4), \quad F(A \otimes B) = Q^\dagger (A \otimes B) Q,$$

with some unitary matrix $Q \in U(4)$. As far as we know this is the first time that the map was realized by the adjoint action. See also [5], where a slightly different matrix R has been used

in place of Q. This R (in our notation) is interesting enough and is called the magic matrix by Makhlin, see also [6] and [7].

As the Hamiltonian H above is real symmetric ($\in su(4)$) (not anti–symmetric ($\in so(4)$)) it is not obvious whether the magic matrix R could be applied to it or not. As we will show presently, a fairly wide sub-class of such Hamiltonians can be explicitly exponentiated with the help of the magic matrix and an additional similarity transformation. For the generic Hamiltonian H, we provide an approximate result of explicit exponentiation.

Here we note that there is some overlap between our work and [7]. However, the methods given in [7] are quite varied and rather complicated. On the other hand ours rely on a consistent use of the magic matrix, so we believe our method could be easily digested by general readers, though the scope might be limited.

As a closely related subject, we discuss the Baker-Campbell-Hausdorff (B-C-H) formula. It is one of the fundamental theorems in elementary Linear Algebra (or Lie group):

$$\mathrm{e}^{A}\mathrm{e}^{B} = \mathrm{e}^{BCH(A,B)} \ ; \ BCH(A,B) = A + B + \frac{1}{2}[A,B] + \frac{1}{12}\{[[A,B],B] + [A,[A,B]]\} + \cdots,$$

where A and B are elements of some algebra. See for example the textbooks [8], [9] or [10]. A closed expression for the B-C-H formula for $SU(2)$ is quite well-known. Similar closed expression is obtained for $SO(4)$ (namely, for $A, B \in so(4)$) by making use of the magic matrix, see [2].

Here we address the problem of explicit summation of the right hand side of the B-C-H formula for certain lower dimensional matrices, in particular, those related to $su(4)$.

In this paper we treat matrices of type

$$H = \begin{pmatrix} 0 & h_{12} & 0 & h_{14} \\ h_{12} & 0 & h_{23} & 0 \\ 0 & h_{23} & 0 & h_{34} \\ h_{14} & 0 & h_{34} & 0 \end{pmatrix},$$

which are very important in quantum optics, quantum computation, etc. (see [1]). We will derive its exact exponential form e^{-itH} and will present a closed form expression of the B-C-H formula for two of them.

2 Magic Matrix

In this section we introduce appropriate concepts and notation together with a brief review of the results in [5] within our necessity, which look slightly different from the original work of Makhlin in [4].

The 1-qubit space is $\mathbf{C}^2 = \mathrm{Vect}_{\mathbf{C}}\{|0\rangle, |1\rangle\}$ where

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1}$$

Let $\{\sigma_1, \sigma_2, \sigma_3\}$ be the Pauli matrices acting on the space

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

Next let us consider the 2–qubit space. Now we use the tensor product notation which is different from the usual one. That is,

$$\mathbf{C}^2 \otimes \mathbf{C}^2 = \{a \otimes b \mid a, b \in \mathbf{C}^2\}, \ \mathbf{C}^2 \widehat{\otimes} \mathbf{C}^2 = \left\{ \sum_{j=1}^{k} \lambda_j a_j \otimes b_j \mid a_j, b_j \in \mathbf{C}^2, \ \lambda_j \in \mathbf{C}, \ k \in \mathbf{N} \right\} \cong \mathbf{C}^4.$$

Then the 2-qubit space is

$$\mathbf{C}^2 \widehat{\otimes} \mathbf{C}^2 = \mathrm{Vect}_{\mathbf{C}}\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\},$$

where $|ab\rangle = |a\rangle \otimes |b\rangle$ $(a, b \in \{0, 1\})$. The Bell basis $\{|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle, |\Psi_4\rangle\}$ defined by

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \ |\Psi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \ |\Psi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \ |\Psi_4\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{3}$$

plays an important role in various context.

By $H_0(2; \mathbf{C})$ we denote the set of all traceless hermitian matrices in $M(2; \mathbf{C})$. It is well-known that they are spanned by the Pauli matrices

$$H_0(2; \mathbf{C}) = \{\mathbf{a} \equiv a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 \mid a_1, a_2, a_3 \in \mathbf{R}\}$$

and $H_0(2; \mathbf{C}) \cong su(2)$ where $su(2) = \mathfrak{L}(SU(2))$ is the Lie algebra of the group $SU(2)$.

The isomorphism is realized as the adjoint action (the Makhlin's theorem) as follows

$$F : SU(2) \otimes SU(2) \longrightarrow SO(4), \quad F(A \otimes B) = R^{\dagger}(A \otimes B)R$$

where

$$R \equiv (|\Psi_1\rangle, -i|\Psi_2\rangle, -|\Psi_3\rangle, -i|\Psi_4\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & -i & -1 & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & 0 & i \end{pmatrix}. \tag{4}$$

Note that the unitary matrix R is a bit different from Q in [4].

Let us consider this isomorphism at the Lie algebra level because it is in general easier than at the Lie group level:

$$
\begin{tikzcd}
\mathfrak{L}(SU(2)\otimes SU(2)) \arrow[r, "f"] \arrow[d, "\exp"'] & \mathfrak{L}(SO(4)) \arrow[d, "\exp"] \\
SU(2)\otimes SU(2) \arrow[r, "F"'] & SO(4)
\end{tikzcd}
$$

Since the Lie algebra of $SU(2)\otimes SU(2)$ is

$$\mathfrak{L}(SU(2)\otimes SU(2)) = su(2)\otimes 1_2 + 1_2 \otimes su(2) = \{i(\mathbf{a}\otimes 1_2 + 1_2\otimes \mathbf{b}) \mid \mathbf{a},\ \mathbf{b}\in H_0(2;\mathbf{C})\}\,,$$

we have only to examine

$$f(i(\mathbf{a}\otimes 1_2 + 1_2\otimes \mathbf{b})) = iR^\dagger(\mathbf{a}\otimes 1_2 + 1_2\otimes \mathbf{b})R \in \mathfrak{L}(SO(4)) \equiv so(4). \tag{5}$$

If we set $\mathbf{a} = \sum_{j=1}^{3} a_j\sigma_j$ and $\mathbf{b} = \sum_{j=1}^{3} b_j\sigma_j$ then the right hand side of (5) reads

$$iR^\dagger(\mathbf{a}\otimes 1_2 + 1_2\otimes \mathbf{b})R = \begin{pmatrix} 0 & a_1+b_1 & a_2-b_2 & a_3+b_3 \\ -(a_1+b_1) & 0 & a_3-b_3 & -(a_2+b_2) \\ -(a_2-b_2) & -(a_3-b_3) & 0 & a_1-b_1 \\ -(a_3+b_3) & a_2+b_2 & -(a_1-b_1) & 0 \end{pmatrix}. \tag{6}$$

Conversely, if

$$A = \begin{pmatrix} 0 & f_{12} & f_{13} & f_{14} \\ -f_{12} & 0 & f_{23} & f_{24} \\ -f_{13} & -f_{23} & 0 & f_{34} \\ -f_{14} & -f_{24} & -f_{34} & 0 \end{pmatrix} \in so(4)$$

then we obtain

$$RAR^\dagger = i(\mathbf{a}\otimes 1_2 + 1_2\otimes \mathbf{b}) \tag{7}$$

with

$$\mathbf{a} = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 = \frac{f_{12}+f_{34}}{2}\sigma_1 + \frac{f_{13}-f_{24}}{2}\sigma_2 + \frac{f_{14}+f_{23}}{2}\sigma_3, \tag{8}$$

$$\mathbf{b} = b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3 = \frac{f_{12}-f_{34}}{2}\sigma_1 - \frac{f_{13}+f_{24}}{2}\sigma_2 + \frac{f_{14}-f_{23}}{2}\sigma_3. \tag{9}$$

It is very interesting to note that $\mathbf{a}$ and $\mathbf{b}$ are the self-dual and anti-self-dual part of the matrix A, respectively, under the Hodge $*$-operation defined by $(*F)_{ij} = \frac{1}{2}\sum_{k,l=1}^{4}\epsilon_{ijkl}F_{kl}$. Here ϵ_{ijkl} is the totally anti-symmetric tensor with $\epsilon_{1234}=1$.

3 The Exponential of Matrices in Four Level System

This section provides the main results of the paper, explicit calculation of the exponential of certain matrices. The generic Hamiltonian treated in this paper is a real symmetric 4×4 matrix with vanishing diagonal elements

$$H=\begin{pmatrix} 0 & h_{12} & h_{13} & h_{14} \\ h_{12} & 0 & h_{23} & h_{24} \\ h_{13} & h_{23} & 0 & h_{34} \\ h_{14} & h_{24} & h_{34} & 0 \end{pmatrix}, \qquad h_{ij}\in\mathbf{R}. \tag{10}$$

The general method is quite simple: evaluate the evolution operator (matrix) $U(t)\equiv \mathrm{e}^{-itH}$ by unitary conjugation of the Hamiltonian H in terms of the magic matrix R:

$$U(t)=\mathrm{e}^{-itH}=RR^{\dagger}\mathrm{e}^{-itH}RR^{\dagger}=R\mathrm{e}^{-itR^{\dagger}HR}R^{\dagger}. \tag{11}$$

The conjugated Hamiltonian $R^{\dagger}HR$ reads explicitly

$$R^{\dagger}HR=\frac{1}{2}\times$$
$$\begin{pmatrix} 2h_{14} & -i(h_{12}+h_{13}+h_{24}+h_{34}) & -h_{12}+h_{13}-h_{24}+h_{34} & 0 \\ i(h_{12}+h_{13}+h_{24}+h_{34}) & 2h_{23} & 0 & h_{12}+h_{13}-h_{24}-h_{34} \\ -h_{12}+h_{13}-h_{24}+h_{34} & 0 & -2h_{23} & i(h_{12}-h_{13}-h_{24}+h_{34}) \\ 0 & h_{12}+h_{13}-h_{24}-h_{34} & -i(h_{12}-h_{13}-h_{24}+h_{34}) & -2h_{14} \end{pmatrix}, \tag{12}$$

whose structure is better displayed in the following tensor product notation

$$\begin{aligned} R^{\dagger}HR=&\left(\frac{h_{13}-h_{24}}{2}\sigma_1+\frac{h_{14}+h_{23}}{2}\sigma_3\right)\otimes 1_2+1_2\otimes\left(\frac{h_{13}+h_{24}}{2}\sigma_2+\frac{h_{14}-h_{23}}{2}\sigma_3\right) \\ &+\frac{h_{12}+h_{34}}{2}\sigma_3\otimes\sigma_2-\frac{h_{12}-h_{34}}{2}\sigma_1\otimes\sigma_3. \end{aligned} \tag{13}$$

Obviously the first two terms commute with each other and the last two terms $\sigma_3\otimes\sigma_2$ and $\sigma_1\otimes\sigma_3$ also commute. The right hand side of (13) is a summation of two blocks not commuting each other.

A few comments are in order.
(1) In the calculation above we have used $R^{\dagger}HR$ in stead of $RHR^{\dagger}$ in (7). This point is important (we leave it to the readers to contemplate why it is so).
(2) The space of the entangled states in the two-qubit system is $SU(4)/SU(2)\otimes SU(2)$, which is identified with the homogeneous space $SU(4)/SO(4)$ because of the isomorphism $SU(2)\otimes SU(2)\cong SO(4)$. Then it is well-known that

$$\left\{A\in SU(4)\mid A^{T}=A\right\}=\left\{A^{T}A\mid A\in SU(4)\right\}\cong SU(4)/SO(4),$$

see for example [5]. By use of the expression $SU(4) \ni A = \mathrm{e}^{iK}$ with $K \in H_0(4;\mathbf{C})$ we have

$$SU(4)/SO(4) \cong \left\{ \mathrm{e}^{iK} \mid K \in H_0(4;\mathbf{R}) \right\},$$

where $H_0(4;\mathbf{R})$ is the set of all traceless symmetric matrices in $M(4;\mathbf{R})$. It is of dimension 9 (see (14)), as expected: $15(su(4)) - 6(so(4)) = 9$.

Since this K is written as

$$K = \begin{pmatrix} h_1 & h_{12} & h_{13} & h_{14} \\ h_{12} & h_2 & h_{23} & h_{24} \\ h_{13} & h_{23} & h_3 & h_{34} \\ h_{14} & h_{24} & h_{34} & h_4 \end{pmatrix} = \mathrm{diag}(h_1,h_2,h_3,h_4) + H, \quad h_1+h_2+h_3+h_4 = 0, \tag{14}$$

the evaluation of the evolution operator $U(t) = \mathrm{e}^{-itH}$ is deeply related with the study of the entangled states in the two-qubit systems.

For later convenience the conjugated matrix of K is given here:

$$\begin{aligned} R^\dagger K R =& \left(\frac{h_{13}-h_{24}}{2}\sigma_1 + \frac{h_{14}+h_{23}}{2}\sigma_3 \right) \otimes 1_2 + 1_2 \otimes \left(\frac{h_{13}+h_{24}}{2}\sigma_2 + \frac{h_{14}-h_{23}}{2}\sigma_3 \right) \\ &+ \frac{h_{12}+h_{34}}{2}\sigma_3 \otimes \sigma_2 - \frac{h_{12}-h_{34}}{2}\sigma_1 \otimes \sigma_3 \\ &+ \frac{h_0-h_1-h_2+h_3}{4}\sigma_3 \otimes \sigma_3 + \frac{h_0+h_1-h_2-h_3}{4}\sigma_1 \otimes \sigma_2 + \frac{h_0-h_1+h_2-h_3}{4}\sigma_2 \otimes \sigma_1. \end{aligned} \tag{15}$$

3.1 Approximate Result

We do not know yet how to exponentiate the generic form of the (R-conjugated) Hamiltonian (13). Our first main result is an approximate one for the generic case:

$$U(t) \approx U_1(t)U_2(t)U_3(t)U_4(t), \tag{16}$$

where each factor in (13) can be exponentiated exactly,

$$U_1(t) = R\left\{ \exp\left(-it\left(\frac{h_{13}-h_{24}}{2}\sigma_1 + \frac{h_{14}+h_{23}}{2}\sigma_3 \right) \right) \otimes 1_2 \right\} R^\dagger, \tag{17}$$

$$U_2(t) = R\left\{ 1_2 \otimes \exp\left(-it\left(\frac{h_{13}+h_{24}}{2}\sigma_2 + \frac{h_{14}-h_{23}}{2}\sigma_3 \right) \right) \right\} R^\dagger, \tag{18}$$

$$U_3(t) = R\exp\left(-it\left(\frac{h_{12}+h_{34}}{2}\sigma_3 \otimes \sigma_2 \right) \right) R^\dagger, \tag{19}$$

$$U_4(t) = R\exp\left(-it\left(-\frac{h_{12}-h_{34}}{2}\sigma_1 \otimes \sigma_3 \right) \right) R^\dagger. \tag{20}$$

As remarked earlier, U_1 and U_2 commute with each other and U_3 and U_4 also commute but the other pairs do not commute. Hopefully the approximation is not so bad. The non-exactness arises from the non-commutativity.

3.2 Special Exact Result I

If $h_{12} = h_{34} = 0$ in (13) the troublesome non-commutativity disappears. Then we have the simple form

$$R^\dagger H R = \left(\frac{h_{13}-h_{24}}{2}\sigma_1 + \frac{h_{14}+h_{23}}{2}\sigma_3\right) \otimes 1_2 + 1_2 \otimes \left(\frac{h_{13}+h_{24}}{2}\sigma_2 + \frac{h_{14}-h_{23}}{2}\sigma_3\right) \tag{21}$$

with

$$H = \begin{pmatrix} 0 & 0 & h_{13} & h_{14} \\ 0 & 0 & h_{23} & h_{24} \\ h_{13} & h_{23} & 0 & 0 \\ h_{14} & h_{24} & 0 & 0 \end{pmatrix}. \tag{22}$$

Then we have the exact form

$$U(t) = U_1(t)U_2(t) = U_2(t)U_1(t) \tag{23}$$

with $U_1(t)$ in (17) and $U_2(t)$ in (18).

Though this Hamiltonian is of very special form there is some application shown in the following section.

3.3 Special Exact Result II

The Hamiltonian that we really want to study is

$$H = \begin{pmatrix} 0 & h_{12} & 0 & h_{14} \\ h_{12} & 0 & h_{23} & 0 \\ 0 & h_{23} & 0 & h_{34} \\ h_{14} & 0 & h_{34} & 0 \end{pmatrix}, \tag{24}$$

which is more general than the one discussed in recent papers by the author [1] [2] and [3], which is obtained by setting $h_{14} = 0$. This restricted case was also discussed in [7]. At first sight this Hamiltonian looks rather different from (22), which can be exactly exponentiated. However, it can be reduced to the form of (22) by a similarity transformation in terms of the exchange (swap) matrix S:

$$SHS = \begin{pmatrix} 0 & 0 & h_{12} & h_{14} \\ 0 & 0 & h_{23} & h_{34} \\ h_{12} & h_{23} & 0 & 0 \\ h_{14} & h_{34} & 0 & 0 \end{pmatrix}, \tag{25}$$

with

$$S = \frac{1}{2}(1_2 \otimes 1_2 + \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad S = S^T = S^{-1}.$$

Therefore we have

$$R^\dagger SHSR = \left(\frac{h_{12} - h_{34}}{2}\sigma_1 + \frac{h_{14} + h_{23}}{2}\sigma_3 \right) \otimes 1_2 + 1_2 \otimes \left(\frac{h_{12} + h_{34}}{2}\sigma_2 + \frac{h_{14} - h_{23}}{2}\sigma_3 \right). \quad (26)$$

The evolution operator

$$U(t) = \mathrm{e}^{-itH} = SRR^\dagger S \mathrm{e}^{-itH} SRR^\dagger S = SR\mathrm{e}^{-itR^\dagger SHSR} R^\dagger S$$

takes an exact factorized form

$$U(t) = U_1(t)U_2(t) = U_2(t)U_1(t) \quad (27)$$

where

$$U_1(t) = SR \left\{ \exp\left(-it \left(\frac{h_{12} - h_{34}}{2}\sigma_1 + \frac{h_{14} + h_{23}}{2}\sigma_3 \right) \right) \otimes 1_2 \right\} R^\dagger S, \quad (28)$$

$$U_2(t) = SR \left\{ 1_2 \otimes \exp\left(-it \left(\frac{h_{12} + h_{34}}{2}\sigma_2 + \frac{h_{14} - h_{23}}{2}\sigma_3 \right) \right) \right\} R^\dagger S. \quad (29)$$

Obviously the two factors commute and they belong to two independent $SU(2)$. Compare this result with that of [7].

4 B-C-H Formula for a class of matrices in SU(4)

In this section we give a closed expression to the B-C-H formula for the set of unitary matrices $\{\mathrm{e}^{iH} \mid H \text{ is a type of (24)}\}$ by use of the results in the preceding two sections. Let us prepare some notation for simplicity.

For two (real) symmetric matrices $A,\ B$

$$A = \begin{pmatrix} 0 & f_1 & 0 & f_4 \\ f_1 & 0 & f_2 & 0 \\ 0 & f_2 & 0 & f_3 \\ f_4 & 0 & f_3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & g_1 & 0 & g_4 \\ g_1 & 0 & g_2 & 0 \\ 0 & g_2 & 0 & g_3 \\ g_4 & 0 & g_3 & 0 \end{pmatrix} \quad (30)$$

we can set

$$R^\dagger(SAS)R = \mathbf{a}_1 \otimes 1_2 + 1_2 \otimes \mathbf{a}_2, \quad R^\dagger(SBS)R = \mathbf{b}_1 \otimes 1_2 + 1_2 \otimes \mathbf{b}_2.$$

Let us represent these four $su(2)$ elements in terms of the $\mathbf{R^3}$ vectors as in (35):

$$\mathbf{a}_1 = \frac{f_1 - f_3}{2}\sigma_1 + \frac{f_4 + f_2}{2}\sigma_3, \quad \mathbf{a}_2 = \frac{f_1 + f_3}{2}\sigma_2 + \frac{f_4 - f_2}{2}\sigma_3,$$
$$\mathbf{b}_1 = \frac{g_1 - g_3}{2}\sigma_1 + \frac{g_4 + g_2}{2}\sigma_3, \quad \mathbf{b}_2 = \frac{g_1 + g_3}{2}\sigma_2 + \frac{g_4 - g_2}{2}\sigma_3$$

with

$$\vec{\mathbf{a}}_1 = \begin{pmatrix} \frac{f_1-f_3}{2} \\ 0 \\ \frac{f_4+f_2}{2} \end{pmatrix}, \quad \vec{\mathbf{b}}_1 = \begin{pmatrix} \frac{g_1-g_3}{2} \\ 0 \\ \frac{g_4+g_2}{2} \end{pmatrix}; \qquad \vec{\mathbf{a}}_2 = \begin{pmatrix} 0 \\ \frac{f_1+f_3}{2} \\ \frac{f_4-f_2}{2} \end{pmatrix}, \quad \vec{\mathbf{b}}_2 = \begin{pmatrix} 0 \\ \frac{g_1+g_3}{2} \\ \frac{g_4-g_2}{2} \end{pmatrix}.$$

Obviously $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{b}}_1$ belong to the same $su(2)$, whereas $\vec{\mathbf{a}}_2$ and $\vec{\mathbf{b}}_2$ belong to the other $su(2)$. Next we introduce a pair of three real parameters $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ defined for each pair of $\mathbf{R^3}$ vectors $(\vec{\mathbf{a}}_1, \vec{\mathbf{b}}_1)$ and $(\vec{\mathbf{a}}_2, \vec{\mathbf{b}}_2)$:

$$\alpha_1 = \alpha(\vec{\mathbf{a}}_1, \vec{\mathbf{b}}_1), \quad \beta_1 = \beta(\vec{\mathbf{a}}_1, \vec{\mathbf{b}}_1), \quad \gamma_1 = \gamma(\vec{\mathbf{a}}_1, \vec{\mathbf{b}}_1),$$
$$\alpha_2 = \alpha(\vec{\mathbf{a}}_2, \vec{\mathbf{b}}_2), \quad \beta_2 = \beta(\vec{\mathbf{a}}_2, \vec{\mathbf{b}}_2), \quad \gamma_2 = \gamma(\vec{\mathbf{a}}_2, \vec{\mathbf{b}}_2).$$

See (37) in the Appendix or the paper [11] for the definition of these parameters. By combining the exact B-C-H formula for $su(2)$ for each pair $(\vec{\mathbf{a}}_1, \vec{\mathbf{b}}_1)$ and $(\vec{\mathbf{a}}_2, \vec{\mathbf{b}}_2)$, we obtain

$$\begin{aligned}
\mathrm{e}^{iA}\mathrm{e}^{iB} &= SRR^\dagger S \mathrm{e}^{iA} SRR^\dagger S \mathrm{e}^{iB} SRR^\dagger S \\
&= SR\mathrm{e}^{iR^\dagger SASR}\mathrm{e}^{iR^\dagger SBSR}R^\dagger S \\
&= SR\mathrm{e}^{i(\mathbf{a}_1\otimes 1_2 + 1_2\otimes\mathbf{a}_2)}\mathrm{e}^{i(\mathbf{b}_1\otimes 1_2 + 1_2\otimes\mathbf{b}_2)}R^\dagger S \\
&= SR\left(\mathrm{e}^{i\mathbf{a}_1}\otimes\mathrm{e}^{i\mathbf{a}_2}\right)\left(\mathrm{e}^{i\mathbf{b}_1}\otimes\mathrm{e}^{i\mathbf{b}_2}\right)R^\dagger S \\
&= SR\left(\mathrm{e}^{i\mathbf{a}_1}\mathrm{e}^{i\mathbf{b}_1}\right)\otimes\left(\mathrm{e}^{i\mathbf{a}_2}\mathrm{e}^{i\mathbf{b}_2}\right)R^\dagger S \\
&= SR\mathrm{e}^{i(\alpha_1\mathbf{a}_1+\beta_1\mathbf{b}_1+\gamma_1\frac{i}{2}[\mathbf{a}_1,\mathbf{b}_1])}\otimes\mathrm{e}^{i(\alpha_2\mathbf{a}_2+\beta_2\mathbf{b}_2+\gamma_2\frac{i}{2}[\mathbf{a}_2,\mathbf{b}_2])}R^\dagger S \\
&= SR\mathrm{e}^{i\left\{(\alpha_1\mathbf{a}_1+\beta_1\mathbf{b}_1+\gamma_1\frac{i}{2}[\mathbf{a}_1,\mathbf{b}_1])\otimes 1_2+1_2\otimes(\alpha_2\mathbf{a}_2+\beta_2\mathbf{b}_2+\gamma_2\frac{i}{2}[\mathbf{a}_2,\mathbf{b}_2])\right\}}R^\dagger S \\
&= \mathrm{e}^{iSR\left\{(\alpha_1\mathbf{a}_1+\beta_1\mathbf{b}_1+\gamma_1\frac{i}{2}[\mathbf{a}_1,\mathbf{b}_1])\otimes 1_2+1_2\otimes(\alpha_2\mathbf{a}_2+\beta_2\mathbf{b}_2+\gamma_2\frac{i}{2}[\mathbf{a}_2,\mathbf{b}_2])\right\}R^\dagger S} \\
&\equiv \mathrm{e}^{iBCH(A,B)}.
\end{aligned} \tag{31}$$

The desired closed form of the B-C-H formula reads

$$\begin{aligned}
&BCH(A,B) \\
&= SR\left\{\left(\alpha_1\mathbf{a}_1+\beta_1\mathbf{b}_1+\gamma_1\frac{i}{2}[\mathbf{a}_1,\mathbf{b}_1]\right)\otimes 1_2 + 1_2\otimes\left(\alpha_2\mathbf{a}_2+\beta_2\mathbf{b}_2+\gamma_2\frac{i}{2}[\mathbf{a}_2,\mathbf{b}_2]\right)\right\}R^\dagger S \\
&= \begin{pmatrix} 0 & (12) & (13) & (14) \\ \overline{(12)} & 0 & (23) & (24) \\ \overline{(13)} & \overline{(23)} & 0 & (34) \\ \overline{(14)} & \overline{(24)} & \overline{(34)} & 0 \end{pmatrix},
\end{aligned} \tag{32}$$

whose entries are

$$
\begin{aligned}
(12) &= \alpha_1 \frac{f_1-f_3}{2} + \beta_1 \frac{g_1-g_3}{2} + \alpha_2 \frac{f_1+f_3}{2} + \beta_2 \frac{g_1+g_3}{2}, \\
(13) &= i\left\{\gamma_1\left(\frac{f_1-f_3}{2}\frac{g_4+g_2}{2} - \frac{f_4+f_2}{2}\frac{g_1-g_3}{2}\right) - \gamma_2\left(\frac{f_1+f_3}{2}\frac{g_4-g_2}{2} - \frac{f_4-f_2}{2}\frac{g_1+g_3}{2}\right)\right\}, \\
(14) &= \alpha_1 \frac{f_4+f_2}{2} + \beta_1 \frac{g_4+g_2}{2} + \alpha_2 \frac{f_4-f_2}{2} + \beta_2 \frac{g_4-g_2}{2}, \\
(23) &= \alpha_1 \frac{f_4+f_2}{2} + \beta_1 \frac{g_4+g_2}{2} - \alpha_2 \frac{f_4-f_2}{2} - \beta_2 \frac{g_4-g_2}{2}, \\
(24) &= i\left\{\gamma_1\left(\frac{f_1-f_3}{2}\frac{g_4+g_2}{2} - \frac{f_4+f_2}{2}\frac{g_1-g_3}{2}\right) + \gamma_2\left(\frac{f_1+f_3}{2}\frac{g_4-g_2}{2} - \frac{f_4-f_2}{2}\frac{g_1+g_3}{2}\right)\right\}, \\
(34) &= -\alpha_1 \frac{f_1-f_3}{2} - \beta_1 \frac{g_1-g_3}{2} + \alpha_2 \frac{f_1+f_3}{2} + \beta_2 \frac{g_1+g_3}{2}.
\end{aligned}
$$

Here $\overline{(12)}$ is the complex conjugate of (12).

This is another main result of the present paper.

5 Discussion

In this letter we addressed the problem of explicit exponentiation of the generic Hamiltonian in the general four level system. With the aid of the magic matrix, we obtained an approximate result for the generic case and some exact results for certain restricted forms of the Hamiltonian. It is a good challenge to derive the explicit form of the evolution operator for the most generic Hamiltonian in the four level system.

Acknowledgments
The author wishes to thank K. Funahashi, H. Oike, R. Sasaki and T. Suzuki for their helpful comments and suggestions.

Appendix: B-C-H Formula for SU(2)

In this Appendix we recapitulate the closed form expression of the B-C-H formula for $SU(2)$ in the 2×2 representation, see [11] for more details.

First of all let us recall the well-known exponentiation formula:

$$
\mathrm{e}^{i(x\sigma_1+y\sigma_2+z\sigma_3)} = \cos r 1_2 + \frac{\sin r}{r} i(x\sigma_1+y\sigma_2+z\sigma_3) \tag{33}
$$

$$
= \begin{pmatrix} \cos r + i\frac{\sin r}{r} z & i\frac{\sin r}{r}(x-iy) \\ i\frac{\sin r}{r}(x+iy) & \cos r - i\frac{\sin r}{r} z \end{pmatrix}, \tag{34}
$$

where $r = \sqrt{x^2 + y^2 + z^2}$. This is a simple exercise.

For the group $SU(2)$ it is easy to sum up all terms in the B-C-H expansion by using the above exact exponentiation formula. Because of the obvious relation $su(2) \cong so(3)$, three dimensional vector notation is quite useful for $su(2)$. For 2×2 hermitian matrices,

$$X = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3, \quad Y = y_1\sigma_1 + y_2\sigma_2 + y_3\sigma_3 \in H_0(2, \mathbf{C})$$

we associate $\mathbf{R^3}$ vectors. Namely, we set

$$X \longrightarrow \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad Y \longrightarrow \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \tag{35}$$

and

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3 = \mathrm{Tr}(XY)/2, \quad |\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}, \quad |\mathbf{y}| = \sqrt{\mathbf{y} \cdot \mathbf{y}}.$$

Then the commutator of X and Y corresponds to the vector product $\mathbf{x} \times \mathbf{y}$:

$$-\frac{i}{2}[X, Y] \longrightarrow \mathbf{x} \times \mathbf{y} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}.$$

Now we are in a position to state the B-C-H formula for $SU(2)$:

$$\mathrm{e}^{iX}\mathrm{e}^{iY} = \mathrm{e}^{iZ} \quad : \quad Z = \alpha X + \beta Y + \gamma\frac{i}{2}[X, Y], \tag{36}$$

where the real coefficients α, β and γ are defined by

$$\begin{aligned} &\alpha \equiv \alpha(\mathbf{x}, \mathbf{y}) = \frac{\sin^{-1}\rho}{\rho}\frac{\sin|\mathbf{x}|\cos|\mathbf{y}|}{|\mathbf{x}|}, \quad \beta \equiv \beta(\mathbf{x}, \mathbf{y}) = \frac{\sin^{-1}\rho}{\rho}\frac{\cos|\mathbf{x}|\sin|\mathbf{y}|}{|\mathbf{y}|}, \\ &\gamma \equiv \gamma(\mathbf{x}, \mathbf{y}) = \frac{\sin^{-1}\rho}{\rho}\frac{\sin|\mathbf{x}|\sin|\mathbf{y}|}{|\mathbf{x}||\mathbf{y}|} \end{aligned} \tag{37}$$

with

$$\begin{aligned} \rho^2 &\equiv \rho(\mathbf{x}, \mathbf{y})^2 \\ &= \sin^2|\mathbf{x}|\cos^2|\mathbf{y}| + \sin^2|\mathbf{y}| \\ &\qquad -\frac{\sin^2|\mathbf{x}|\sin^2|\mathbf{y}|}{|\mathbf{x}|^2|\mathbf{y}|^2}(\mathbf{x} \cdot \mathbf{y})^2 + \frac{2\sin|\mathbf{x}|\cos|\mathbf{x}|\sin|\mathbf{y}|\cos|\mathbf{y}|}{|\mathbf{x}||\mathbf{y}|}(\mathbf{x} \cdot \mathbf{y}). \end{aligned}$$

The proof is not difficult, so it is left to the readers.

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